MTH 301: Group Theory

Homework III

(Due 16/09)

1. Let G be a group, $H \leq G$, and $N \trianglelefteq G$. Then show that:
   (a) $NH \leq G$.
   (b) $H \cap N \trianglelefteq H$.
   (c) $N \trianglelefteq NH$.
   (d) If $H \trianglelefteq G$, then $NH \trianglelefteq G$.
   (e) If $o(a)$ is finite for some $a \in G$, then $o(Na) \mid o(a)$.

2. Let G be a group, and $H \leq G$. Then prove that
   (a) $Z(G) \trianglelefteq G$.
   (b) $N(H) \leq G$.
   (c) $H \trianglelefteq N(H)$.
   (d) $N(H)$ is the largest subgroup in which H is normal.
   (e) $H \trianglelefteq G$ if, and only if $N(H) = G$.

3. Let G be a group, and let $g \in G$ be a fixed element. Then show that the map $\varphi_g : G \to G$ defined by
$$\varphi_g(x) = gxg^{-1}, \text{ for all } x \in G,$$
is an isomorphism.

4. For a group G, consider the subgroup generated $[G, G] = \langle S \rangle$ generated by elements in the set
$$S = \{ghg^{-1}h^{-1} \mid g, h \in G\}.$$
   (a) Show that $[G, G] \trianglelefteq G$. [This subgroup is called the *commutator subgroup or the derived subgroup* of G. It is also denoted by G' or $G^{(1)}$.]
   (b) Show that $G/[G, G]$ is abelian.
   (c) Show that G is abelain if, and only if $[G, G] = \{1\}$.

5. Using the First Isomorphism Theorem, establish the following isomophisms.
   (a) $\mathrm{GL}(n, \mathbb{C})/\mathrm{SL}(n, \mathbb{C}) \cong \mathbb{C}^{\times}$
   (b) $S_n/A_n \cong \mathbb{Z}_2$.
   (c) $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.
   (d) $\mathbb{C}^{\times}/N = \mathbb{R}^{+}$, where $N = \{z \in \mathbb{C} \mid |z| = 1\}$ and $\mathbb{R}^{+}$ is the group of positive reals under multiplication.